6
Poincaré-based control of delayed measured systems: Limitations and Improved Control[1]

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6.1 Introduction

What is the effect of measurement delay on Ott, Grebogi, and Yorke (OGY) chaos control? Which possibilities exists for improved control? These questions are addressed within this chapter, and the OGY control case is considered as well as a related control scheme, difference control; both together form the two main Poincaré-based chaos control schemes, where the control amplitude is computed once during the orbit after crossing the Poincaré section.

If the stabilization of unstable periodic orbits or fixed points by the method given by Ott, Grebogi, Yorke [23] and Hübler [15] can only be based on a measurement delayed by τ orbit lengths, resulting in a control loop latency, the performance of unmodified OGY control is expected to decay. For experimental considerations, it is desired to know the range of stability with minimal knowledge of the system. In section 6.3, the area of stability is investigated both for OGY control and for difference control, yielding a delay-dependent maximal Lyapunov number beyond which control fails. Sections 6.3.4 to 6.4.3 address the question how the control of delayed measured chaotic systems can be improved, i.e., what extensions must be considered if one wants to stabilize fixed points with a higher Lyapunov number. Fortunately, the limitation can be overcome most elegantly by linear control methods that employ memory terms, as linear predictive logging control (Sec. 6.4.1) and memory difference control (Sec. 6.4.3). In both cases, one is equipped with an explicit deadbeat control scheme that allows, within linear approximation, to perform control without principal limitations in delay time, dimension, and Ljapunov numbers.

6.1.1 The delay problem - time-discrete case

For fixed point stabilization in time-continuous control, the issue of delay has been investigated widely in control theory, dating back at least to the Smith predictor [31].

[1]To appear in: E. Schöll & H. G. Schuster (Eds.), Handbook of Chaos Control, Wiley-VCH (2007).

This approach mimics the, yet unknown, actual system state by a linear prediction based on the last measurement. Its time-discrete counterparts discussed in this chapter allow to place all eigenvalues of the associated linear dynamics to zero, and always ensure stability. The (time-continuous) Smith predictor with its infinite-dimensional initial condition had to be refined [24, 12], giving rise to the recently active fields of *model predictive control* [3]. For fixed point stabilization, an extension of permissible latency has been found for a modified proportional-plus-derivative controller [28].

Delay is also a generic problem in the control of chaotic systems. The effective delay time τ in any feedback loop is the sum of at least three delay times, the duration of measurement, the time needed to compute the appropriate control amplitude, and the response time of the system to the applied control. The latter effect appears especially when the applied control additionally has to propagate through the system. These response time may extend to one or more cycle lengths [21]. If one wants to stabilize the dynamics of a chaotic system onto an unstable periodic orbit, one is in a special situation. In principle, a proper engineering approach could be to use the concept of sliding mode control [10], i.e. to use a co-moving coordinate system and perform suitable control methods within it. However, this requires quite accurate knowledge of whole trajectory and stable manifold, with respective numerical or experimental costs.

Therefore direct approaches have been developed by explicitely taking into account either a Poincaré surface of section [23] or the explicit periodic orbit length [26]. This field of *controlling chaos*, or stabilization of chaotic systems, by small perturbations, in system variables [15] or control parameters [23], has developed to a widely discussed topic with applications in a broad area from technical to biological systems. Especially in fast systems [29, 2] or for slow drift in parameters [4, 22], difference control methods have been successful, namely the time-continuous Pyragas scheme [26], ETDAS [29], and time-discrete difference control [1].

Like for the control method itself, the discussion of the measurement delay problem in chaos control has to take into account the special issues of the situation: In classical control applications one always tries to keep the control loop latency as short as possible. In chaotic systems however, one wants to control a fixed point of the Poincaré iteration and thus has to wait until the next crossing of the Poincaré surface of section, where the system again is in vicinity of that fixed point.

The stability theory and the delay influence for time-continuous chaos control schemes has been studied extensively [17, 19, 11, 18, 14], and an improvement of control by periodic modulation has been proposed in [20]. For measurement delays that extend to a full period, however no extension of the time-continuous Pyragas scheme is available.

This chapter discusses the major Poincaré-based control schemes OGY control [23] and difference feedback [1] in the presence of time delay, and addresses the question what strategies can be used to overcome the limitations due to the delay [8]. We show how the measurement delay problem can be solved systematically for OGY control and difference control by rhythmic control and memory methods and give constructive direct and elegant formulas for the deadbeat control in the time-discrete Poincaré iteration. While the predictive control method LPLC presented below for

OGY control has a direct correspondence to the Smith predictor and thus can be reviewed as its somehow straightforward implementation within the unstable subspace of the Poincaré iteration, this prediction approach does not guarantee a stable controller for difference control. However, within a class of feedback schemes linear in system parameters and system variable, there is always a unique scheme where all eigenvalues are zero, i. e. the MDC scheme presented below. The method can be applied also for more than one positive Ljapunov exponent, and shows, within validity of the linearization in vicinity of the orbit, to be free of principal limitations in Ljapunov exponents or delay time. For zero delay (but the inherent period one delay of difference control), MDC has been demonstrated experimentally for a chaotic electronic circuit [4] and a thermionic plasma discharge diode [22], with excellent agreement, both of stability areas and transient Ljapunov exponents, to the theory presented here. This chapter is organized as follows. After introducing the notation within a recall of OGY control, we give a brief summary what limitations occur for unmodified OGY control; details can be found in [8]. From Section 6.3.6 we introduce different memory methods to improve control, of which the LPLC approach appears to be superior as it allows stabilization of arbitrary fixed points for any given delay. The stabilization of unknown fixed points is discussed in Section 6.4.3, where we present a memory method (MDC) that again allows stabilization of arbitrary unstable fixed points. For all systems with only one instable Lyapunov number, the iterated dynamics can be transformed on an eigensystem which reduces to the one-dimensional case, and the generalization to the case of higher-dimensional subspaces is straightforward [9].

6.1.2 Experimental setups with delay

Before discussing the time-discrete reduced dynamics in the Poincaré iteration, it should be clarified how this relates to an experimental control situation. On a first glance, the time-discrete viewpoint seems to correspond only to a case where the delay (plus waiting time to the next Poincaré section) exactly matches the orbit length, or a multiple of it. Generically, in the experiment one experiences a non-matching delay. Application of all control methods discussed here requires to introduce an additional delay, usually by waiting for the next Poincaré crossing, so that measurement and control are applied without phase shift at the same position of the orbit. In this case the next Poincaré crossing position x_{t+1} is a function of the values of x and r at a finite number of previous Poincaré crossings only, i. e. it does not depend on intermediate positions. Therefore the (a priori infinite-dimensional) delay system reduces to a finite-dimensional iterated map. If the delay (plus the time of the waiting mechanism to the next Poincaré crossing) does not match the orbit length, the control schemes may perform less efficient. Even for larger deviations from the orbit, the time between the Poincaré crossings will vary only marginally, thus a control amplitude should be available in time. In practical situations therefore the delay should not exceed the orbit length minus the variance of the orbit length that appears in the respective system and control setup.

In a formal sense, the Poincaré approach ensures robustness with respect to uncer-

tainties in the orbit length, as it always ensures a synchronized reset of both trajectories and control. Between the Poincaré crossings the control parameter is constant, the system is independent of everything *in advance of* the last Poincaré crossing. It is solely determined by the differential equation (or experimental dynamics). Thus the next crossing position is a well defined iterated function of the previous one. This is quite in contrast to the situation of a delay-differential equation (as in Pyragas control), which has an infinite-dimensional initial condition it 'never gets rid of'. One may proceed to stability analysis via Floquet theory [13] as investgated for continuous [17] and impulse length issues in Poincaré-based [5, 6, 7] control schemes. Though a Poincaré crossing detection may be applied as well, the position will depend not only on the last crossing, but also on all values of the system variable within a time horizon defined by the maximum of the delay length and the (maximal) time difference between two Poincaré crossings (being non-stroboscopic). Thus the Poincaré iteration would be a function between two infinite-dynamical spaces. Contrary to a delay differential equation with *fixed* delay, a major advantage of a Poincaré map is to reduce the system dynamics to a low-dimensional system; therefore *for all control schemes discussed here,* the additional dimensionality is not a continuous horizon of states, but merely a finite set of values that were measured at the previous Poincaré crossings.

6.2 Ott – Grebogi – Yorke (OGY) control

The method of Ott, Grebogi and Yorke [23] stabilizes unstable fixed points, or unstable periodic orbits utilizing a Poincaré surface of section, by feedback that is applied in vicinity of the fixed point x^* of a discrete dynamics $x_{t+1} = f(x_t, r)$. For a chaotic flow, or corresponding experiment, the system dynamics $\dot{\vec{x}} = \vec{F}(\vec{x}, r)$ reduces to the discrete dynamics between subsequent Poincaré sections at $t_0, t_1, \ldots t_n$. This description is fundamentally different from a stroboscopic sampling as long as the system is not on a periodic orbit, where the sequence of differences $(t_i - t_{i-1})$ would show a periodic structure.

If there is only one positive Ljapunov exponent, we can proceed considering the motion in unstable direction only. One can transform on the eigensystem of the Jacobi matrix $\frac{\partial f}{\partial r}$ and finds again the equations of the one-dimensional case, i.e. one only needs to apply control in the unstable direction (see e.g. [5, 9]). Thus stabiliy analysis and control schemes of the one-dimensional case holds also for higher-dimensional systems provided there is only one unstable direction. For two or more positive Ljapunov exponents on can proceed in a similar fashion [5, 9].

In OGY control, the control parameter r_t is made time-dependent. The amplitude of the feedback $r_t = r - r_0$ added to the control parameter r_0 is proportional by a constant ε to the distance $x - x^*$ from the fixed point, i. e. $r = r_0 + \varepsilon(x_t - x^*)$, and the feedback gain can be determined from a linearization around the fixed point, which reads, if we neglect higher order terms,

$$f(x_t, r_o + r_t) \;\; = \;\; f(x^*, r_0) + (x_t - x^*) \cdot \left(\frac{\partial f}{\partial x}\right)_{x^*, r_0}$$

$$\begin{aligned} & +r_t \cdot \left(\frac{\partial f}{\partial r}\right)_{x^*,r_0} \\ = \;& f(x^*, r_0) + \lambda(x_t - x^*) + \mu r_t \\ = \;& f(x^*, r_0) + (\lambda + \mu\varepsilon) \cdot (x_t - x^*) \end{aligned} \tag{6.1}$$

The second expression vanishes for $\varepsilon = -\lambda/\mu$, that is, in linear approximation the system arrives at the fixed point at the next time step, $x_{t+1} = x^*$. The uncontrolled system is assumed to be unstable in the fixed point, i. e. $|\lambda| > 1$. The system with applied control is stable if the absolute value of the eigenvalues of the iterated map is smaller than one,

$$|x_{t+1} - x^*| = |(\lambda + \mu\varepsilon) \cdot (x_t - x^*)| < |x_t - x^*| \tag{6.2}$$

Therefore ε has to be chosen between $(-1-\lambda)/\mu$ and $(+1-\lambda)/\mu$, and this interval is of width $2/\mu$ and independent of λ, i.e. fixed points with arbitrary λ can be stabilized. This property however does not survive for delayed measurement. [8], as surveyed below.

6.3 Limitations of unmodified control and simple improved control schemes

In this section the limitations of unmodified control are discussed, both for OGY control and for difference control. For completeness, rhythmic control and a state space memory control are discussed in Sections 6.3.4 and 6.3.6.

6.3.1 Limitations of unmodified OGY control in presence of delay

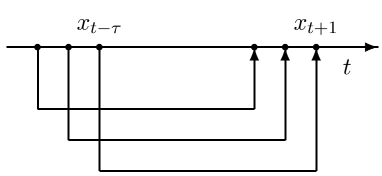

Figure 6.1: Unmodified control in the presence of delay (schematically).

We want to know what limitations occur if the OGY rule is applied without modification. Intuitively, one expects the possibility of unstable behaviour of $(\tau+1)$ control loops that mutually overlap in the course of time (see Figure 6.1). In OGY control, the control parameter r_t is time-dependent, and without loss of generality we assume that $x^* = 0$ and that $r_t = 0$ if no control is applied. First we discuss the simplest relevant case $\tau = 1$ explicitely. For one time step delay, instead of $r_t = \varepsilon x_t$ we have the proportional feedback rule:

$$r_t = \varepsilon x_{t-1}. \tag{6.3}$$

Using the time-delayed coordinates (x_t, x_{t-1}), the linearized dynamics of the system with applied control is given by $\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} = \begin{pmatrix} \lambda & \mu\varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix}.$

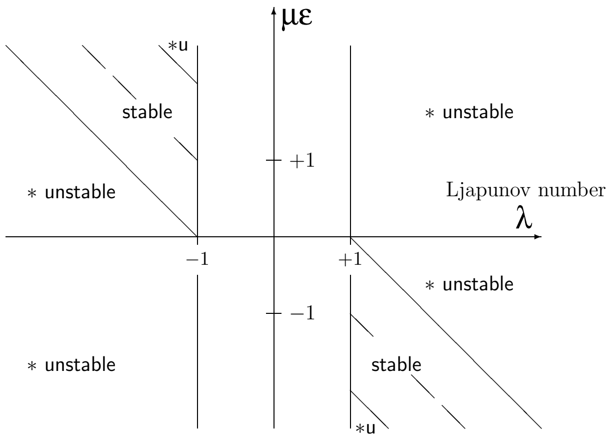

Figure 6.2: Stability range of OGY control

The eigenvalues of $\begin{pmatrix} \lambda & \mu\varepsilon \\ 1 & 0 \end{pmatrix}$ are given by $\alpha_{1,2} = \frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} + \varepsilon\mu}$. Control can be achieved with ε being in an interval $]-1/\mu, (1-\lambda)/\mu[$ with the width $(2-\lambda)/\mu$ (see Figure 6.2).

In contrast to the not-delayed case, we have a requirement $\lambda < 2$ for the Lyapunov number: Direct application of the OGY method fails for systems with a Lyapunov number of 2 and higher [4, 8]. This limitation is caused by the additional degree of freedom introduced in the system due to the time delay.

Now we consider the general case. If the system is measured delayed by τ steps, $r_t = \varepsilon x_{t-\tau}$, we can write the dynamics in time-delayed coordinates $(x_t, x_{t-1}, x_{t-2}, \ldots x_{t-\tau})^{\mathrm{T}}$:

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-\tau+1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & & \cdots & 0 & \varepsilon\mu \\ 1 & 0 & & & & & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & & \ddots & & \\ \vdots & & & & \ddots & 0 & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-\tau} \end{pmatrix} \tag{6.4}$$

The characteristic polynomial is given by (we define rescaled coordinates $\tilde{\alpha} := \alpha/\lambda$ and $\tilde{\varepsilon} = \varepsilon\mu/\lambda^{\tau+1}$)

$$\begin{aligned} 0 &= P(\alpha) = (\alpha - \lambda)\alpha^\tau - \varepsilon\mu \\ \text{or} \qquad 0 &= P(\tilde{\alpha}) = (\tilde{\alpha} - 1)\tilde{\alpha}^\tau - \tilde{\varepsilon}. \end{aligned} \tag{6.5}$$

Fig. 6.3 shows the maximum of the absolute value of the eigenvalues. In rescaled coordinates $\tilde{\alpha} = 1/\lambda$ corresponds to a control interval $\tilde{\varepsilon}_\pm(\tau, \lambda)$. For

$$\lambda_{\max} = 1 + \frac{1}{\tau} \tag{6.6}$$

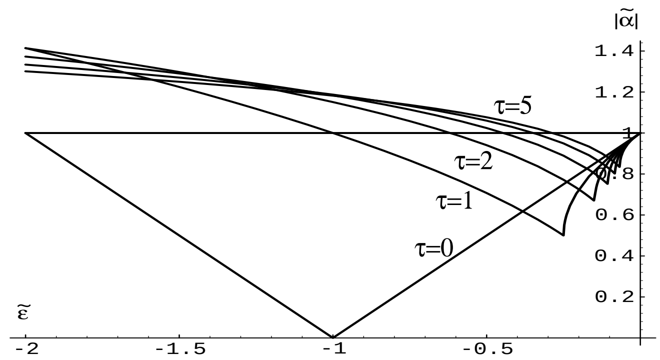

Figure 6.3: Control intervals for several time delays $\tau = 0 \ldots 5$: The plots show the maximal absolute value of the eigenvalues as a function of the rescaled control gain $\tilde{\varepsilon}$. Values of $|\tilde{\alpha}| = 1/\lambda$ correspond to $|\alpha| = 1$ in (6.5) without rescaling, so one can obtain the range $]\varepsilon_-, \varepsilon_+[$ for which control is successfully achieved.

the control interval vanishes, and for $\lambda \geq \lambda_{\max}(\tau)$ no control is possible [4, 8]. If we look at the Lyapunov exponent $\Lambda := \ln \lambda$ instead of the Lyapunov number, we find with $\ln x < (x-1)$ the inequality

$$\Lambda_{\max} \cdot \tau < 1. \tag{6.7}$$

Therefore, delay time and Lyapunov exponent limit each other if the system is to be controlled. This is consistent with the loss of knowledge in the system by exponential separation of trajectories.

6.3.2 Stability diagrams derived by the Jury criterion

For small τ one can derive easily the borders of the stability area with help of the Jury criterion [5, 8]. The Jury criterion [16] gives a sufficient and necessary condition that all roots of a given polynomial are of modulus smaller than unity. Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, one applies the iterative *Jury table* scheme:

$$\begin{array}{rrcl} \forall_{0 \leq i \leq n} & b_i & := & a_{n-i} \\ & \alpha_n & := & b_n / a_n \\ \forall_{1 \leq i \leq n} & a_{i-1}^{\text{new}} & := & a_i - \alpha_n b_i \end{array}$$

giving α_n and coefficients $a_{n-1} \ldots a_0$ for the next iteration. The Jury criterion states that the eigenvalues are of modulus smaller than unity if and only if $\forall_{1 \leq i \leq n} |\alpha_i| < 1$. The criterion gives $2n$ (usually partly redundant) inequalities that define hypersurfaces in coefficient space. The complete set of lines is shown in Figure 6.4 for $\tau = 4$ to illustrate the redundancy of the inequalities generated by the Jury table. For $\tau = 1$,

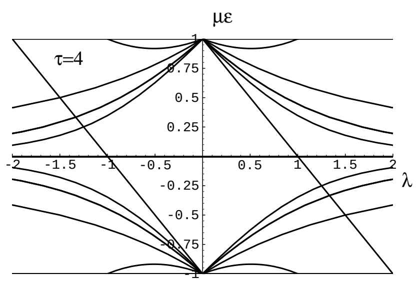

Figure 6.4: Complete Jury diagram for $\tau = 4$.

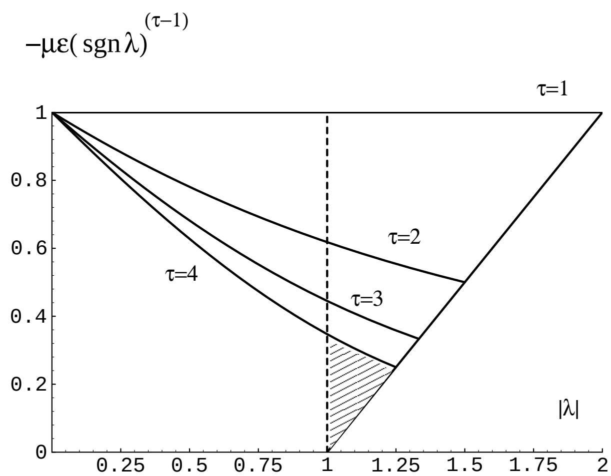

Figure 6.5: Stability areas for $\tau = 1, 2, 3, 4$, combined. Only for $|\lambda| > 1$ control is necessary (dashed line), and the stability area (shaded for $\tau = 4$) extends to $|\lambda_{\max}| = 2,\ 3/2,\ 4/3,\ 5/4$. Note that still both positive and negative λ can be controlled. The abscissa $-\mu\varepsilon(\mathrm{sgn}\lambda)^{(\tau-1)}$ takes into account that for odd τ a negative $\mu\varepsilon$ is required, independent of the sign of λ.

the Jury coefficients are given by $\alpha_1 = -\lambda/(1+\mu\varepsilon)$ and $\alpha_2 = -\mu\varepsilon$. Control is only necessary for $|\lambda| > 1$, and by folding the relevant stability area into the same quadrant one obtains Fig. 6.5 showing how $\lambda_{\max}$ decreases for increasing τ.

6.3.3 Stabilizing unknown fixed points: Limitiations of unmodified difference control

As the OGY approach discussed above requires the knowledge of the position of the fixed point, one may wish to stabilize purely by feeding back differences of the system variable at different times. This becomes relevant in the case of parameter drifts [4] which often can occur in experimental situations. A time-continuous strategy $r(t) = \varepsilon(x(t) - x(t - \tau_{\mathsf{d}}))$ has been introduced by Pyragas [26], where $r(t)$ is updated continuously and τ_{d} matches the period of the unstable periodic orbit. The time-discrete counterpart (i.e. control amplitudes are calculated every Poincaré section) is the difference control scheme [1]: For control without delay, a simple difference control strategy

$$r_t = \varepsilon(x_{t-\tau} - x_{t-\tau-1}) \tag{6.8}$$

is possible for $\varepsilon\mu = -\lambda/3$, and eigenvalues of modulus smaller than unity of the matrix $\begin{pmatrix} \lambda+\varepsilon\mu & -\varepsilon\mu \\ 1 & 0 \end{pmatrix}$ are obtained only for $-3 < \lambda < +1$, so this method stabilizes only for oscillatory repulsive fixed points with $-3 < \lambda < -1$ [1], see the $\tau = 0$ case in Figure 6.6).

We can proceed in a similar fashion as for OGY control. In the presence of τ steps delay the linearized dynamics of difference control is given by

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-\tau} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & & 0 & \varepsilon\mu & -\varepsilon\mu \\ 1 & 0 & & & & & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & & \ddots & & \\ \vdots & & & & \ddots & 0 & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-\tau-1} \end{pmatrix}$$

in delayed coordinates $(x_t, x_{t-1}, \ldots x_{t-\tau-1})$, and the characteristic polynomial is given by

$$0 = (\alpha - \lambda)\alpha^{\tau+1} + (1-\alpha)\varepsilon\mu. \tag{6.9}$$

As we have to use $x_{t-\tau-1}$ in addition to $x_{t-\tau}$, the system is of dimension $\tau + 2$, and the lower bound of Lyapunov numbers that can be controlled are found to be

$$\lambda_{\text{inf}} = -\frac{3+2\tau}{1+2\tau} = -\left(1 + \frac{1}{\tau + 1/2}\right) \tag{6.10}$$

and the asymptotic control amplitude at this point is

$$\varepsilon\mu = \frac{(-1)^\tau}{1+2\tau}. \tag{6.11}$$

The stability area in the $(\mu\varepsilon, \lambda)$ plane is bounded by the lines $\alpha_i = \pm 1$ where α_i are the coefficients given by the Jury criterion [16] (see Figure 6.6). For $\tau = 0$, the Jury coefficients are $\alpha_1 = -\frac{\lambda+\varepsilon\mu}{1+\varepsilon\mu}$ and $\alpha_2 = \varepsilon\mu$. For $\tau = 1$ to $\tau = 3$, the Jury coefficients are given in [8].

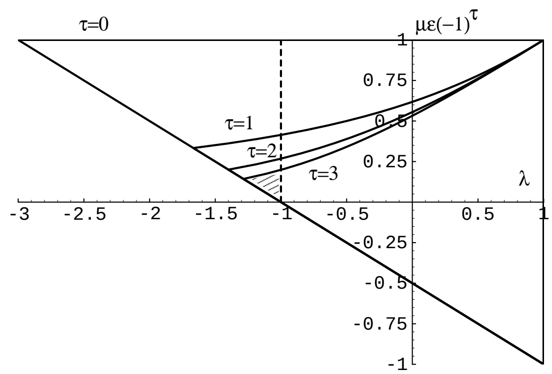

Figure 6.6: Difference feedback for $\tau = 0, 1, 2, 3$: Stability borders derived by the Jury criterion [5, 8]. The stability diagram of the non-delayed case $\tau = 0$ has already been given in [1]. From $\lambda = -1$ (dashed line) to $\lambda = +1$ the system ist stable without control. For each τ, control is effective only within the respective area (shaded for $\tau = 3$).

The controllable range is smaller than for the unmodified OGY method, and is restricted to oscillatory repulsive fixed points with $\lambda_{\text{inf}} < \lambda \leq -1$. Thus, delay severely reduces the number of controllable fixed points, and one has to develop special control strategies for the control of delayed measured systems. A striking observation is that inserting $\tau + \frac{1}{2}$ for τ in eq. (6.6) exactly leads to the expression in eq. (6.10) which reflects the fact that the difference feedback control can be interpreted as a discrete first derivative, taken at time $t - (\tau + \frac{1}{2})$. Thus the controllability relation (6.7) holds again.

As λ^{-1} is implying a natural time scale (that of exponential separation) of an orbit, it is quite natural that control becomes limited by a border proportional to a product of λ and a feedback delay time. Already without the additional difficulty of a measurement delay this is expected to appear for any control scheme that itself is using time-delayed feedback: E.g. the extensions of time-discrete control schemes

discussed in [30] with an inherent Lyapunov number limitation due to memory terms, and the experimentally widely applied time-continuous schemes Pyragas and ETDAS [17, 19, 11]. Here Pyragas control has the Lyapunov exponent limitation $\Lambda\tau_{\mathsf{p}} \leq 2$ together with the requirement of the Floquet multiplier of the uncontrolled orbit having an imaginary part of π, meaning that deviations from the orbit after one period experience to be flipped around the orbit by that angle, which is quite the generic case [18]. This nicely corresponds with the requirement of a negative Lyapunov number that appears in difference control. A positive Lyapunov number in the time-discrete picture corresponds to a zero flip of the time-continuous orbit, and is consistently uncontrollable in both schemes.

Recently, the influence of a control loop latency has also been studied for continuous time-delayed feedback [18] by Floquet analysis, obtaining a critical value for the measurement delay τ, that corresponds to a maximal Lyapunov exponent $\log|\lambda_{\mathsf{inf}}| = \Lambda\tau_{\mathsf{p}} = \frac{1}{1/2+\tau/\tau_{\mathsf{p}}}$, where τ_{p} is the orbit length and matched feedback delay. By the log inequality that again translates (for small Ljapunov exponents) to our result for the time-discrete difference control. An exact coincidence could not be expected, as in Pyragas control the feedback difference is computed continuously sliding with the motion along the orbit, where in difference control it is evaluated within each Poincaré section. For the ETDAS scheme with latency, a detailed analysis is performed in [14], showing that the range of stability can be extended compared to the Pyragas scheme. Although the time-continuous case (as an a priori infinite-dimensional delay-differential system) could exhibit much more complex behaviour, it is, however astonishing that for all three methods, OGY, difference, and Pyragas control, the influence of measurement delay mainly results in the same limitation of the controllable Lyapunov number.

6.3.4 Rhythmic control schemes: Rhythmic OGY control

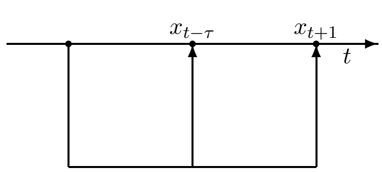

Figure 6.7: Rhythmic control (schematically). Keeping control quiet for τ intermediate time steps avoids the additional degrees of freedom. However, the effective Lyapunov number to be controlled then is raised to $\lambda^{\tau+1}$.

As pointed out for difference control in the case $\tau = 0$ in [1], one can eliminate the additional degrees of freedom caused by the delay term. One can restrict himself to apply control rhythmically only every $\tau+1$ timesteps ($\tau+2$ for difference control), and then leave the system uncontrolled for the remaining timesteps (see Fig. 6.7). Then $\varepsilon = \varepsilon(t)$

appears to be time-dependent with

$$\varepsilon(t \bmod \tau) = (\varepsilon_0, 0, \ldots, 0) \tag{6.12}$$

and, after $(\tau + 1)$ iterations of (6.4), we again have a matrix as in (6.4), but with $\lambda^{\tau+1}$ instead of λ. Equivalently, we can write

$$x_{t+(\tau+1)} = \lambda^{\tau+1} x_t + \varepsilon_0 \mu x_t. \tag{6.13}$$

What we have done here, is: controlling the $(\tau + 1)$-fold iterate of the original system. This appears to be formally elegant, but leads to practically uncontrollable high effective Lyapunov numbers $\lambda^{\tau+1}$ for both large λ and large τ.

Even if the rhythmic control method is of striking simplicity, it remains unsatisfying that control is kept quiet, or inactive, for τ time steps. Even if the state of the system x is known delayed by τ, one knows (in principle) the values of x_t for $t < \tau$, and one could (in principle) store the values $\delta r_{t-\tau} \ldots \delta r_t$ of the control amplitudes applied to the system. This can be done, depending on the timescale, by analog or digital delay lines, or by storing the values in a computer or signal processor (observe that there are some intermediate frequency ranges where an experimental setup is difficult).

Both methods, rhythmic control and simple feedback control in every time step, have their disadvantages: For rhythmic control it is necessary to use rather large control amplitudes, in average λ^τ/τ, and noise sums up to an amplitude increased by factor $\sqrt{\tau}$. For simple feedback control the dimension of the system is increased and the maximal controllable Lyapunov number is bounded by (6.6). One might wonder if there are control strategies that avoid these limitations. This has necessarily to be done by applying control in each time step, but with using knowledge what control has been applied between the last measured time step $t - \tau$ and t. This concept can be implemented in at least two ways, by storing previous values of x_t (Section 6.3.6) or previous values of δr_t (LPLC, Section 6.4.1 and MDC, Section 6.4.3).

6.3.5 Rhythmic difference control

To enlarge the range of controllable λ, one again has the possibility to reduce the dimension of the control process in linear approximation to one by applying control every $\tau + 2$ time steps.

$$\begin{aligned} x_{t+1} &= \lambda x_t + \mu\varepsilon(x_{t-\tau} - x_{t-\tau-1}) \\ &= (\lambda^{\tau+1} + \mu\varepsilon\lambda - \mu\varepsilon) x_{t-\tau-1} \end{aligned} \tag{6.14}$$

and the goal $x_{t+1} \stackrel{!}{=} 0$ can be fulfilled by

$$\mu\varepsilon = -\frac{\lambda^{\tau+1}}{1 - \lambda} \tag{6.15}$$

One has to choose $\mu\varepsilon$ between $\mu\varepsilon_{\pm} = -\frac{\lambda^{\tau+1}\pm 1}{1-\lambda}$ to achieve control as shown in Fig. 6.8. The case $\tau = 0$ has already been discussed in [1]. With rhythmic control, there is no range limit for λ, and even fixed points with positive λ can be stabilized by this method.

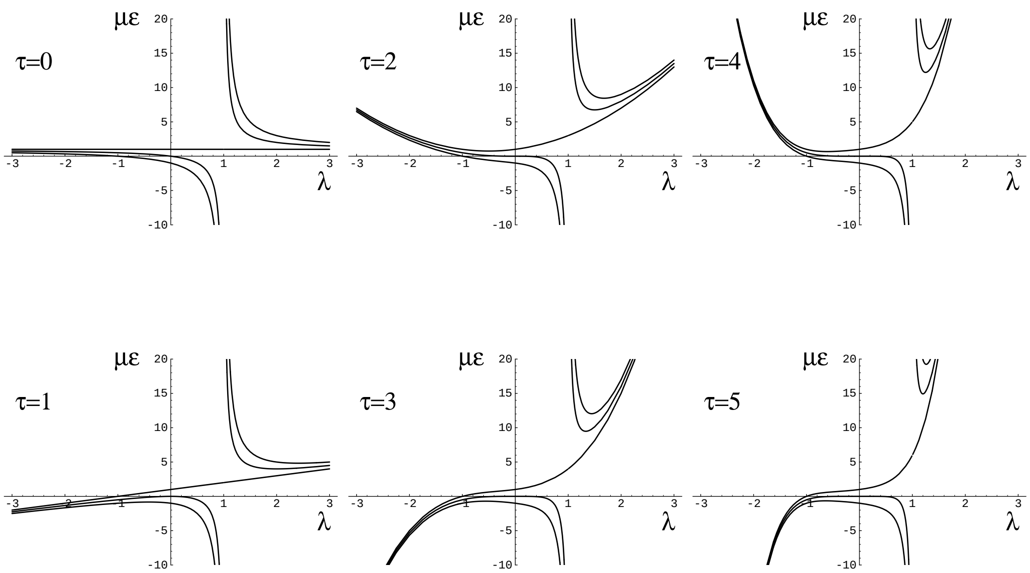

Figure 6.8: Stability area of rhythmic difference control for $\tau = 0, 1, 2, 3, 4, 5$.

When using differences for periodic feedback, one still has the problem that the control gain increases by λ^{τ}, and noise sums up for $\tau+1$ time steps before the next control signal is applied. Additionally, now there is a singularity for $\lambda = +1$ in the "optimal" control gain given by (6.14). This concerns fixed points where differences $x_t - x_{t-1}$ when escaping from the fixed point are naturally small due to a λ near to $+1$.

Here one has to decide between using a large control gain (but magnifying noise and finite precision effects) or using a small control gain of order $\mu\varepsilon_{-}(\lambda = +1) = \tau + 1$ (but having larger eigenvalues and therefore slow convergence).

Two other strategies that have been discussed by Socolar and Gauthier [30] are discretized versions of time-continuous methods. Control between $\lambda = -(3+R)/(1-R)$ and $\lambda = -1$ is possible with discrete-ETDAS ($R < 1$) $r_t = \varepsilon \sum_{k=0}^{\infty} R^k (x_{t-k} - x_{t-k-1})$ and control between $\lambda = -(N+1)$ and $\lambda = -1$ is acheived with discrete-NTDAS (let N be a positive integer) which is defined by $r_t = \varepsilon \left(x_t - \frac{1}{N} \sum_{k=0}^{N} x_{t-k} \right)$. Both methods can be considered to be of advantage even in time-discrete control in the Poincaré section, e.g. if the number of adjustable parameters has to be kept small. Whereas these methods are mainly applied in time-continuous control, especially in analogue or optical experiments, for time-discrete control the MDC strategy described below allows to overcome the Lyapunov number limitations.

6.3.6 A simple memory control scheme: Using state space memory

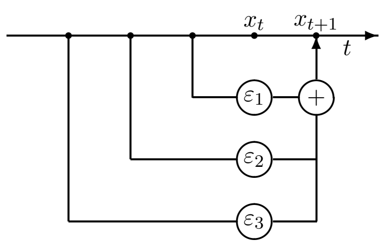

Figure 6.9: A state space memory control (schematically). For electronic or optic analogue circuits, the idea to use additional delay lines is appealing, though the applicability is restricted to the $\tau = 1$ OGY case (which will cover most experiments).

We extend the single delay line by several artificial delay lines (see Fig. 6.9), each with an externally tuneable control gain coefficient [5, 9]:

$$r_t = \varepsilon_1 x_{t-1} + \varepsilon_2 x_{t-2} + \ldots + \varepsilon_{n+1} x_{t-n-1} \tag{6.16}$$

For n steps memory (and one step delay) the control matrix is

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-n} \end{pmatrix} = \begin{pmatrix} \lambda & \varepsilon_1 & \cdots & & & \varepsilon_n & \varepsilon_{n+1} \\ 1 & 0 & & & & & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & & \ddots & & \\ \vdots & & & & \ddots & 0 & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \\ \\ \\ \vdots \\ x_{t-n-1} \end{pmatrix} \tag{6.17}$$

with the characteristic polynomial $(\alpha - \lambda)\alpha^{n+1} + \sum_{i=1}^{n} \varepsilon_i \alpha^{n-i}$. We can choose $\alpha_1 = \alpha_2 = \ldots \alpha_{n+2} = -\lambda/(n+2)$ and evaluate optimal values for all ε_i by comparing with the coefficients of the product $\prod_{i=1}^{n+2}(\alpha - \alpha_i)$. This method allows control up to $\lambda_{\max} = 2 + n$, thus arbitrary λ can be controlled if a memory length of $n > \lambda - 2$ and the optimal coefficents ε_i are used.

For more than one step delay, one has the situation $\varepsilon_1 = 0, \ldots, \varepsilon_{\tau-1} = 0$. This prohibits the 'trivial pole placement' given above, (choosing all α_i to the same value) and therefore reduces the maximal controllable λ and no general scheme for optimal selection of the ε_i applies. One can alternatively use the LPLC method described below, which provides an optimal control scheme. One could wonder why to consider the previous state memory scheme at all when it does not allow to make all eigenvalues zero in *any* case. First, the case of up to one orbit delay and moderately small λ already covers many low-period orbits. Second, there may be experimental setups where the feedback of previous states through additional delay elements and an analog circuit is experimentally more feasible than feedback of past control amplitudes.

6.4 Optimal Improved Control schemes

6.4.1 Linear predictive logging control (LPLC)

If it is possible to store the previously applied control amplitudes $r_t, r_{t-1}, \ldots$, then one can predict the actual state x_t of the system using the linear approximation around the fixed point (see Fig. 6.10). That is, from the last measured value $x_{t-\tau}$ and the control amplitudes we compute estimated values iteratively by

$$y_{t-i+1} = \lambda x_{t-i} + \mu r_{t-i} \tag{6.18}$$

leading to a *predicted* value y_t of the actual system state. Then the original OGY formula can be applied, i. e. $r_t = -\lambda/\mu y_t$. In this method the gain parameters are again linear in $x_{t-\tau}$ and all $\{r_{t'}\}$ with $t - \tau \leq t' \leq t$, and the optimal gain parameters can be expressed in terms of λ and μ.

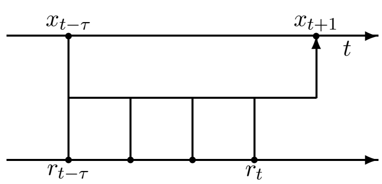

Figure 6.10: Linear predictive logging control (schematically). In LPLC, all intermediately applied control amplitudes are employed for a linear prediction. A corresponding scheme (MDC, Section 6.4.3) exists also for difference control.

In contrast to the memory method presented in the previous subsection, the LPLC method directs the system (in linear approximation) in one time step onto the fixed point. However, when this control algorithm is switched on, one has no control applied between $t-\tau$ and $t-1$, so the trajectory has to be fairly near to the orbit (in an interval with a length of order δ/λ^τ, where δ is the interval halfwidth where control is switched on). Therefore the time one has to wait until the control can be successfully activated is of order $\lambda^{\tau-1}$ larger than in the case of undelayed control.

The LPLC method can also be derived as a general linear feedback in the last measured system state and all applied control amplitudes since the system was measured – by choosing the feedback gain parameters in a way that the linearized system has all eigenvalues zero. The linear ansatz

$$r_t = \varepsilon \cdot x_{t-\tau-i} + \eta_1 r_{t-1} + \ldots \eta_\tau r_{t-\tau} \tag{6.19}$$

leads to the dynamics in combined delayed coordinates
$(x_t, x_{t-1}, \ldots, x_{t-\tau}, r_{t-1}, \ldots, r_{t-\tau})$

$$
\begin{pmatrix} x_{t+1} \\ x_t \\ \\ \vdots \\ \\ x_{t-\tau+1} \\ r_t \\ \vdots \\ \\ r_{t-\tau+1} \end{pmatrix}
=
\begin{pmatrix}
\lambda & 0 & \cdots & \cdots & 0 & \varepsilon & \eta_1 & \eta_2 & \cdots & \cdots & \eta_\tau \\
 & 0 & & & & & & & & & \\
 & 1 & \ddots & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & & & & & & \\
 & & & & 1 & 0 & & & & & \\
0 & 0 & \cdots & \cdots & 0 & \varepsilon & \eta_1 & \eta_2 & \cdots & \cdots & \eta_\tau \\
 & & & & & & 1 & 0 & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & 1 & 0
\end{pmatrix}
\begin{pmatrix} x_t \\ x_{t-1} \\ \\ \vdots \\ \\ x_{t-\tau} \\ r_{t-1} \\ \vdots \\ \\ r_{t-\tau} \end{pmatrix}
$$

giving the characteristic polynomial

$$
\begin{aligned}
0 \quad = \quad & -\alpha^\tau(\alpha^{\tau+1} + \alpha^\tau(-\lambda - \eta_1) \\
& + \alpha^{\tau-1}(\lambda \cdot \eta_1 - \eta_2) + \alpha^{\tau-2}(\lambda \cdot \eta_2 - \eta_3) \\
& \ldots + \alpha^1(\lambda \cdot \eta_{\tau-1} - \eta_\tau) + (\lambda \cdot \eta_\tau - \varepsilon)). \qquad (6.20)
\end{aligned}
$$

All eigenvalues can be set to zero using $\varepsilon = -\lambda^{\tau+1}$ and $\eta_i = -\lambda^i$. A generalization to more than one positive Lyapunov exponent is given in [9].

6.4.2 Nonlinear predictive logging control

One can also consider a nonlinear predictive logging control (NLPLC) strategy [9] as the straightforward extension to the LPLC method for nonlinear prediction. If the system has a delay of several time steps, the interval where control is achieved becomes too small. However, if it is possible to extract the first nonlinearities from the time series, prediction (and control) can be fundamentally improved. In NLPLC, the behaviour of the system is predicted each time step by a truncated Taylor series

$$x_{t+1} = \lambda x_t + \frac{\lambda_2}{2} x_t^2 + \mu r_t + \frac{\mu_2}{2} r_t^2 + \nu x_t r_t + o(x_t^3, x_t^1 r_t, x_t r_t^2, r_t^3)$$

using applied control amplitudes $\{r_t\}$ for each time step. This equation has to be solved for r_t using $x_{t+1} \stackrel{!}{=} 0$. A similar nonlinear prediction method has been described by Petrov and Showalter [25]. They approximate the $x_{t+1}(x_t, r_t)$ surface directly from the time series and use it to direct the system to any desired point. Both Taylor approximation or Petrov and Showalter method can be used here iteratively, provided one knows the delay length. Both approaches could be regarded as a nonlinear method of model predictive control [3], applied to the Poincaré iteration dynamics.

From a practical point of view, it has to be mentioned that one has to know the fixed point x^* more accurate than in the linear case. Otherwise one experiences a smaller range of stability and additionally a permanent nonvanishing control amplitude will remain. This may be of disadvantage especially if the fixed point drifts in time (e.g. by other external parameters such as temperature) or if the time series used to determine the parameters is too short.

6.4.3 Stabilization of unknown fixed points: Memory difference control (MDC)

As all methods mentioned above require the knowledge of the position of the fixed point, one may wish to stabilize purely by feeding back differences of the system variable at different times. Without delay, difference feedback can be used successfully for $\varepsilon\mu = -\lambda/3$, and eigenvalues of modulus smaller than unity of the matrix $\begin{pmatrix} \lambda + \varepsilon\mu & -\varepsilon\mu \\ 1 & 0 \end{pmatrix}$ are obtained only for $-3 < \lambda < +1$, so this method stabilizes only for oscillatory repulsive fixed points with $-3 < \lambda < -1$ [1].

Due to the inherent additional period one delay of difference control and MDC, the τ period delay case of MDC corresponds, in terms of the number of degrees of freedom, to the $\tau + 1$ period delay case of LPLC.

One may wish to generalize the linear predictive feedback to difference feedback. Here, caution is advised. In contrary to the LPLC case, the reconstruction of the state $x_{t-\tau}$ from differences $x_{t-\tau-i} - x_{t-\tau-i-1}$ and applied control amplitudes r_{t-j} is no longer unique. As a consequence, there are infinitely many ways to compute an estimate for the present state of the system, but only a subset of these leads to a controller design ensuring convergence to the fixed point. Among these there exists an optimal every-step control for difference feedback with minimal eigenvalues and in this sense optimal stability.

To derive the feedback rule for MDC [4, 9, 5], we directly make the linear ansatz

$$r_t = \varepsilon \cdot (x_{t-\tau-i} - x_{t-\tau-i-1}) + \eta_1 r_{t-1} + \ldots \eta_\tau r_{t-\tau}$$

with the dynamics in combined delayed coordinates

$$\begin{pmatrix} x_{t+1} \\ x_t \\ \vdots \\ \\ x_{t-\tau+2} \\ x_{t-\tau+1} \\ r_t \\ r_{t-1} \\ \vdots \\ \\ r_{t-\tau+1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & 0 & \varepsilon & -\varepsilon & \eta_1 & \eta_2 - \eta_1 & \cdots & \cdots & \eta_\tau - \eta_{\tau+1} \\ 1 & 0 & & & & & & & & & \\ & 1 & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & 1 & 0 & & & & & \\ 0 & 0 & \cdots & 0 & \varepsilon & -\varepsilon & \eta_1 & \eta_2 - \eta_1 & \cdots & \cdots & \eta_\tau - \eta_{\tau+1} \\ & & & & & & 1 & 0 & & & \\ & & & & & & & \ddots & \ddots & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \\ \vdots \\ \\ x_{t-\tau+1} \\ x_{t-\tau} \\ r_{t-1} \\ r_{t-2} \\ \vdots \\ \\ r_{t-\tau} \end{pmatrix}$$

giving the characteristic polynomial

$$\begin{aligned} 0 &= -\alpha^{\tau}(\alpha^{\tau+1} + \alpha^{\tau}(-\lambda - \eta_1) \\ &\quad +\alpha^{\tau-1}(\lambda \cdot \eta_1 - \eta_2) + \alpha^{\tau-2}(\lambda \cdot \eta_2 - \eta_3) \\ &\quad \ldots + \alpha^{2}(\lambda \cdot \eta_{\tau-2} - \eta_{\tau-1}) \\ &\quad +\alpha^{1}(\lambda \cdot \eta_{\tau-1} - \eta_{\tau} - \varepsilon) + (\lambda \cdot \eta_{\tau} + \varepsilon)). \end{aligned} \tag{6.21}$$

All eigenvalues can be set to zero using $\varepsilon = -\lambda^{\tau+1}/(\lambda - 1)$, $\eta_\tau = +\lambda^{\tau}/(\lambda - 1)$ and $\eta_i = -\lambda^i$ for $1 \leq i \leq \tau - 1$. This defines the MDC method. For more than one positive Lyapunov exponent see [5, 9].

6.5 Summary

Delayed measurement is a generic problem that can appear in controlling chaos experiments. In some situations it may be technically impossible to extend the control method, then one wants to know the stability borders with minimal knowledge of the system.

We have shown that both OGY control and difference control cannot control orbits with an arbitrary Lyapunov number if there is only delayed knowledge of the system. The maximal Lyapunov number up to which an instable orbit can be controlled is given by $1 + \frac{1}{\tau}$ for OGY control and $1 + \frac{1}{\tau+1/2}$ for difference control. For small τ the stability borders can be derived by the Jury criterion, so that the range of values for the control gain ε can be determined from the knowledge of the Taylor coefficients λ and μ. If one wants to overcome these limitations, one has to modify the control strategy.

We have presented methods to improve Poincaré-section based chaos control for delayed measurement. For both classes of algorithms, OGY control and difference control, delay affects control, and improved control strategies have to be applied. Improved strategies contain one of the following principle ideas: Rhythmic control, control with memory for previous states, or control with memory for previously applied control amplitudes. In special cases the unmodified control, previous state memory control, or rhythmic control methods could be considered, especially when experimental conditions restrict the possibilities of designing the control strategy.

In general, the LPLC and MDC strategies allow a so-called deadbeat control with all eigenvalues zero; and they are in this sense optimal control methods. All parameters needed for controller design can be calculated from linearization parameters that can be fitted directly from experimental data. This approach has also been sucessfully applied in an electronic [4] and plasma [22] experiment.

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